

# Positive scalar curvature, higher rho invariants and localization algebras

Zhizhang Xie\*

Department of Mathematics, Vanderbilt University

Guoliang Yu†

Department of Mathematics, Texas A&M University

## Abstract

In this paper, we use localization algebras to study higher rho invariants of closed spin manifolds with positive scalar curvature metrics. The higher rho invariant is a secondary invariant and is closely related to positive scalar curvature problems. The main result of the paper connects the higher index of the Dirac operator on a spin manifold with boundary to the higher rho invariant of the Dirac operator on the boundary, where the boundary is endowed with a positive scalar curvature metric. Our result extends a theorem of Piazza and Schick [21].

## 1 Introduction

Ever since the index theorem of Atiyah and Singer [3], the main goal of index theory has been to associate (topological or geometric) invariants to operators on manifolds. The Atiyah-Singer index theorem states that the analytic index of an elliptic operator on a closed manifold (i.e. compact manifold without boundary) is equal to the topological index of this operator, where the latter can be expressed in terms of topological information of the underlying manifold. The corresponding index theorem for manifolds with boundary was proved by Atiyah, Patodi and Singer [1]. Due to the presence of the boundary, a secondary invariant naturally appears in the index formula. This secondary invariant depends only on the boundary and is called the eta invariant [2].

When one deals with (closed) manifolds that have nontrivial fundamental groups, it turns out that one has a more refined index theory, where the index takes values in the  $K$ -theory group of the reduced  $C^*$ -algebra of the fundamental group. This is

---

\*Email: zhizhang.xie@vanderbilt.edu

†Email: guoliangyu@math.tamu.edu; partially supported by the US National Science Foundation.

now referred to as the higher index theory. One of its main motivations is the Novikov conjecture, which states that higher signatures are homotopy invariants of manifolds. Using techniques from higher index theory to study the Novikov conjecture traces back to the work of Miščenko [20]. With the seminal work of Kasparov [16], and Connes and Moscovici [11], this line of development has turned out to be fruitful and fundamental in the study of the Novikov conjecture. Further development was pursued by many authors, as part of the noncommutative geometry program initiated by Connes [9, 10].

As the higher index theory for closed manifolds has important applications to topology and geometry, one naturally asks whether the same works for manifolds with boundary. In other words, we would like to have a higher version of the Atiyah-Patodi-Singer index theorem for manifolds with boundary. This turns out to be a difficult task and we are still far away from having a complete picture. In particular, a higher version of the eta invariant arises naturally as a secondary invariant. However, we do not know how to define it in general. This higher eta invariant was first studied by Lott [19]. For a fixed closed spin manifold  $M$ , the higher eta invariant is an obstruction for two positive scalar curvature metrics on  $M$  to lie in the same connected component of  $\mathcal{R}^+(M)$  the space of all positive scalar curvature metrics on  $M$ .

The connection of higher index theory to positive scalar curvature problems was already made apparent in the work of Rosenberg (cf. [24]). The applications of the higher eta invariant to the study of positive scalar curvature metrics on manifolds can be found in the work of Leitchnam and Piazza (cf. [18]). The higher rho invariant is a variant of the higher eta invariant. It was also first investigated by Lott in the cyclic cohomological setting [19] (see also Weinberger's paper for a more topological approach [26]). In a recent paper of Piazza and Schick [21], they studied the higher rho invariant on spin manifolds equipped with positive scalar curvature metrics, in the  $K$ -theoretic setting. In particular, their higher rho invariant lies in the  $K$ -theory group of certain  $C^*$ -algebra.

In this paper, we use localization algebras (introduced by the second author [28, 29]) to study the higher rho invariant on spin manifolds equipped with positive scalar curvature metrics. Let  $N$  be a spin manifold with boundary, where the boundary  $\partial N$  is endowed with a positive scalar curvature metric. In the main theorem of this paper, we show that the  $K$ -theoretic “boundary” of the higher index class of the Dirac operator on  $N$  is identical to the higher rho invariant of the Dirac operator on  $\partial N$ . More generally, let  $M$  be a  $m$ -dimensional complete spin manifold with boundary  $\partial M$  such that

- (i) the metric on  $M$  has product structure near  $\partial M$  and its restriction on  $\partial M$ , denoted by  $h$ , has positive scalar curvature;
- (ii) there is an proper and cocompact isometric action of a discrete group  $\Gamma$  on  $M$ ;
- (iii) the action of  $\Gamma$  preserves the spin structure of  $M$ .

We denote the associated Dirac operator on  $M$  by  $D_M$  and the associated Dirac operator on  $\partial M$  by  $D_{\partial M}$ . With the positive scalar curvature metric  $h$  on the boundary  $\partial M$ , we can naturally define the higher index class  $\text{Ind}(D_M)$  of  $D_M$  and the higher rho

invariant  $\rho(D_{\partial M}, h)$  of  $D_{\partial M}$  (see Section 2.2 and 2.3 for the precise definitions). This higher rho invariant  $\rho(D_{\partial M}, h)$  lives in the  $K$ -theory group  $K_{m-1}(C_{L,0}^*(\partial M)^\Gamma)$ . Here  $C_{L,0}^*(\partial M)^\Gamma$  is the kernel of the evaluation map  $\text{ev} : C_L^*(\partial M)^\Gamma \rightarrow C^*(\partial M)^\Gamma$  (see Section 2.2 and 2.3 below for the precise definitions). Notice that the short exact sequence of  $C^*$ -algebras

$$0 \rightarrow C_{L,0}^*(M)^\Gamma \rightarrow C_L^*(M)^\Gamma \rightarrow C^*(M)^\Gamma \rightarrow 0$$

induces the following long exact sequence in  $K$ -theory:

$$\cdots \rightarrow K_i(C_L^*(M)^\Gamma) \rightarrow K_i(C^*(M)^\Gamma) \xrightarrow{\partial_i} K_{i-1}(C_{L,0}^*(M)^\Gamma) \rightarrow K_{i-1}(C_L^*(M)^\Gamma) \rightarrow \cdots$$

We have the following main theorem of the paper.

**Theorem A.**

$$\partial_m(\text{Ind}(D_M)) = \rho(D_{\partial M}, h).$$

In fact, a priori,  $\partial_m(\text{Ind}(D_M))$  and  $\rho(D_{\partial M}, h)$  live in different  $K$ -theory groups. The equality in the theorem above holds only after some natural identifications of various  $K$ -theory groups. We refer to Theorem 22 for the precise statement. This theorem extends a theorem of Piazza and Schick [21, Theorem 1.17] to all dimensions.

As an immediate application, one sees that nonvanishing of the higher rho invariant is an obstruction to extension of the positive scalar curvature metric from the boundary to the whole manifold (cf. Corollary 25 below). Moreover, the higher rho invariant can be used to distinguish whether or not two positive scalar curvature metrics are connected by a path of positive scalar curvature metrics (cf. Corollary 26 below). In a similar context, these have already appeared in the work of Lott [19], Botvinnik and Gilkey [7], and Leichtnam and Piazza [18].

We shall also use the theorem above to map Stolz's positive scalar curvature exact sequence to the exact sequence of Higson and Roe. Recall that Stolz introduced in [25] the following positive scalar curvature exact sequence

$$\longrightarrow \Omega_{n+1}^{\text{spin}}(B\Gamma) \longrightarrow R_{n+1}^{\text{spin}}(B\Gamma) \longrightarrow \text{Pos}_n^{\text{spin}}(B\Gamma) \longrightarrow \Omega_n^{\text{spin}}(B\Gamma) \longrightarrow$$

where  $B\Gamma$  is the classifying space of a discrete group  $\Gamma$ . Moreover,  $\Omega_n^{\text{spin}}(B\Gamma)$  is the spin bordism group,  $\text{Pos}_n^{\text{spin}}(B\Gamma)$  is certain structure group of positive scalar curvature metrics and  $R_{n+1}^{\text{spin}}(B\Gamma)$  is certain obstruction group for the existence of positive scalar curvature metric (see Section 5 below for the precise definitions). This is analogous to the surgery exact sequence in topology. In fact, for the surgery exact sequence, Higson and Roe constructed a natural homomorphism [12, 13, 14] from the surgery exact sequence to the following exact sequence of  $K$ -theory of  $C^*$ -algebras:

$$\rightarrow K_{n+1}(B\Gamma) \rightarrow K_{n+1}(C_r^*(\Gamma)) \rightarrow K_{n+1}^\Gamma(D^*(\Gamma)) \rightarrow K_n(B\Gamma) \rightarrow$$

where  $C_r^*(\Gamma)$  is the reduced  $C^*$ -algebra of  $\Gamma$  and  $D^*(\Gamma) = D^*(X)^\Gamma$  for any proper metric space  $X$  equipped with a proper and cocompact isometric action of  $\Gamma$  (see Section 2.2 for the precise definition). Hence this exact sequence of Higson and Roe

provides natural index theoretic invariants for the surgery exact sequence. Moreover, it is closely related to the Baum-Connes conjecture.

It is a natural task to construct an similar homomorphism from the Stolz's positive scalar curvature exact sequence to the Higson-Roe exact sequence. This was first taken up in a recent paper of Piazza and Schick [21], where they showed that the following diagram commutes when  $n + 1$  is even:

$$\begin{array}{ccccccc}
\longrightarrow & \Omega_{n+1}^{\text{spin}}(B\Gamma) & \longrightarrow & R_{n+1}^{\text{spin}}(B\Gamma) & \longrightarrow & \text{Pos}_n^{\text{spin}}(B\Gamma) & \longrightarrow & \Omega_n^{\text{spin}}(B\Gamma) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_{n+1}(B\Gamma) & \longrightarrow & K_{n+1}(C_r^*(\Gamma)) & \longrightarrow & K_{n+1}^\Gamma(D^*(\Gamma)) & \longrightarrow & K_n(B\Gamma) & \longrightarrow
\end{array}$$

Here all the vertical maps are naturally defined (cf. Section 5 and 6).

Of course, one would expect the above diagram to commute for all dimensions, regardless of the parity of  $n + 1$ . In this paper, we show this is indeed the case. Regarding the method of proof, we use a different approach, which appears to be more conceptual than that of Piazza and Schick. In particular, our proof works equally well for both the even and the odd cases. More precisely, we have the following result.

**Theorem B.** *Let  $X$  be a proper metric space equipped with a proper and cocompact isometric action of a discrete group  $\Gamma$ . For all  $n \in \mathbb{N}$ , the following diagram commutes*

$$\begin{array}{ccccccc}
\Omega_{n+1}^{\text{spin}}(X)^\Gamma & \longrightarrow & R_{n+1}^{\text{spin}}(X)^\Gamma & \longrightarrow & \text{Pos}_n^{\text{spin}}(X)^\Gamma & \longrightarrow & \Omega_n^{\text{spin}}(X)^\Gamma \\
\downarrow \text{Ind}_L & & \downarrow \text{Ind} & & \downarrow \rho & & \downarrow \text{Ind}_L \\
K_{n+1}(C_L^*(X)^\Gamma) & \longrightarrow & K_{n+1}(C^*(X)^\Gamma) & \xrightarrow{\partial} & K_n(C_{L,0}^*(X)^\Gamma) & \longrightarrow & K_n(C_L^*(X)^\Gamma)
\end{array}$$

One of the main concepts that we use is the notion of localization algebras. We refer the reader to Section 5 for the precise definitions of various terms. We point out that the second row of the above diagram is canonically equivalent to the long exact sequence of Higson and Roe (see Proposition 34 below for a proof).

We also remark that  $K_{n+1}(C_L^*(X)^\Gamma) \cong K_{n+1}^\Gamma(X)$ , where the latter group is the equivariant  $K$ -homology with  $\Gamma$ -compact supports of  $X$  [28, Theorem 3.2]. If  $X = \underline{E}\Gamma$  the classifying space of  $\Gamma$ -proper actions, then the map

$$K_{n+1}(C_L^*(X)^\Gamma) \rightarrow K_{n+1}(C^*(X)^\Gamma)$$

in the above commutative diagram can be naturally identified with the Baum-Connes assembly map (cf. [4]). In principle, the higher rho invariant could provide nontrivial secondary invariants that do not lie in the image of the Baum-Connes assembly map.

The paper is organized as follows. In Section 2, we recall the definitions of various basic concepts that will be used later in the paper. In Section 3, we carry out a detailed construction for the higher index class of the Dirac operator on a manifold whose boundary equipped with a positive scalar curvature metric. In Section 4, we prove the main theorem of the paper. In Section 5, we apply our main theorem to map the Stolz's positive scalar curvature exact sequence to a long exact sequence involving

the  $K$ -theory of localization algebras. In Section 6, we show that the  $K$ -theoretic long exact sequence used in Section 5 is canonically isomorphic to the long exact sequence of Higson and Roe. In particular, the explicit construction shows that the definition of the higher rho invariant in our paper is naturally identical to that of Piazza and Schick [21, Section 1.2].

**Acknowledgements** We wish to thank Thomas Schick for bringing the main question of the paper to our attention and for sharing with us a draft of their paper [21].

## 2 Preliminaries

### 2.1 $K$ -theory and index maps

For a short exact sequence of  $C^*$ -algebras

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J} \rightarrow 0,$$

we have a six-term exact sequence in  $K$ -theory:

$$\begin{array}{ccccc} K_0(\mathcal{J}) & \longrightarrow & K_0(\mathcal{A}) & \longrightarrow & K_0(\mathcal{A}/\mathcal{J}) \\ \uparrow \partial_0 & & & & \downarrow \partial_1 \\ K_1(\mathcal{A}/\mathcal{J}) & \longleftarrow & K_1(\mathcal{A}) & \longleftarrow & K_1(\mathcal{J}) \end{array}$$

Let us recall the definition of the boundary maps  $\partial_i$ .

**Even case.** Let  $u$  be an invertible element in  $\mathcal{A}/\mathcal{J}$ . Let  $v$  be the inverse of  $u$  in  $\mathcal{A}/\mathcal{J}$ . Now suppose  $U, V \in \mathcal{A}$  are lifts of  $u$  and  $v$ . We define

$$W = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix} \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that  $W$  is invertible and a direct computation shows that

$$W - \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in \mathcal{J}.$$

Consider the idempotent

$$P = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1} = \begin{pmatrix} UV + UV(1 - UV) & (2 + UV)(1 - UV)U \\ V(1 - UV) & (1 - UV)^2 \end{pmatrix}.$$

We have

$$P - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{J}.$$

By definition,

$$\partial_0([u]) = [P] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(\mathcal{J}).$$

*Remark 1.* If  $u$  is unitary in  $\mathcal{A}/\mathcal{J}$  and  $U \in \mathcal{A}$  is a lift of  $u$ , then we can choose  $V = U^*$ .

**Odd case.** Let  $q$  be an idempotent in  $\mathcal{A}/\mathcal{J}$  and  $Q$  a lift of  $q$  in  $\mathcal{A}$ . Then

$$\partial_1([q]) = [e^{2\pi i Q}] \in K_1(\mathcal{J}).$$

## 2.2 Roe algebras and localization algebras

In this subsection, we briefly recall some standard definitions in coarse geometry. We refer the reader to [22, 28] for more details. Let  $X$  be a proper metric space. That is, every closed ball in  $X$  is compact. An  $X$ -module is a separable Hilbert space equipped with a  $*$ -representation of  $C_0(X)$ , the algebra of all continuous functions on  $X$  which vanish at infinity. An  $X$ -module is called nondegenerate if the  $*$ -representation of  $C_0(X)$  is nondegenerate. An  $X$ -module is said to be standard if no nonzero function in  $C_0(X)$  acts as a compact operator.

**Definition 2.** Let  $H_X$  be a  $X$ -module and  $T$  a bounded linear operator acting on  $H_X$ .

- (i) The propagation of  $T$  is defined to be  $\sup\{d(x, y) \mid (x, y) \in \text{Supp}(T)\}$ , where  $\text{Supp}(T)$  is the complement (in  $X \times X$ ) of the set points  $(x, y) \in X \times X$  for which there exist  $f, g \in C_0(X)$  such that  $gTf = 0$  and  $f(x) \neq 0, g(y) \neq 0$ ;
- (ii)  $T$  is said to be locally compact if  $fT$  and  $Tf$  are compact for all  $f \in C_0(X)$ ;
- (iii)  $T$  is said to be pseudo-local if  $[T, f]$  is compact for all  $f \in C_0(X)$ .

**Definition 3.** Let  $H_X$  be a standard nondegenerate  $X$ -module and  $\mathcal{B}(H_X)$  the set of all bounded linear operators on  $H_X$ .

- (i) The Roe algebra of  $X$ , denoted by  $C^*(X)$ , is the  $C^*$ -algebra generated by all locally compact operators with finite propagations in  $\mathcal{B}(H_X)$ .
- (ii)  $D^*(X)$  is the  $C^*$ -algebra generated by all pseudo-local operators with finite propagations in  $\mathcal{B}(H_X)$ . In particular,  $D^*(X)$  is a subalgebra of the multiplier algebra of  $C^*(X)$ .
- (iii)  $C_L^*(X)$  (resp.  $D_L^*(X)$ ) is the  $C^*$ -algebra generated by all bounded and uniformly continuous functions  $f : [0, \infty) \rightarrow C^*(X)$  (resp.  $f : [0, \infty) \rightarrow D^*(X)$ ) such that

$$\text{propagation of } f(t) \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Again  $D_L^*(X)$  is a subalgebra of the multiplier algebra of  $C_L^*(X)$ .

- (iv)  $C_{L,0}^*(X)$  is the kernel of the evaluation map

$$\text{ev} : C_L^*(X) \rightarrow C^*(X), \quad \text{ev}(f) = f(0).$$

In particular,  $C_{L,0}^*(X)$  is an ideal of  $C_L^*(X)$ . Similarly, we define  $D_{L,0}^*(X)$  as the kernel of the evaluation map from  $D_L^*(X)$  to  $D^*(X)$ .

- (v) If  $Y$  is a subspace of  $X$ , then  $C_L^*(Y; X)$  (resp.  $C_{L,0}^*(Y; X)$ ) is defined to be the closed subalgebra of  $C_L^*(X)$  (resp.  $C_{L,0}^*(X)$ ) generated by all elements  $f$  such that there exist  $c_t > 0$  satisfying  $\lim_{t \rightarrow \infty} c_t = 0$  and  $\text{Supp}(f(t)) \subset \{(x, y) \in X \times X \mid d((x, y), Y \times Y) \leq c_t\}$  for all  $t$ .

Now in addition we assume that a discrete group  $\Gamma$  acts properly and cocompactly on  $X$  by isometries. In particular, if the action of  $\Gamma$  is free, then  $X$  is simply a  $\Gamma$ -covering of the compact space  $X/\Gamma$ .

Now let  $H_X$  be a  $X$ -module equipped with a covariant unitary representation of  $\Gamma$ . If we denote the representation of  $C_0(X)$  by  $\varphi$  and the representation of  $\Gamma$  by  $\pi$ , this means

$$\pi(\gamma)(\varphi(f)\xi) = \varphi(f^\gamma)(\pi(\gamma)\xi),$$

where  $f \in C_0(X)$ ,  $\gamma \in \Gamma$  and  $f^\gamma(x) = f(\gamma^{-1}x)$ . In this case, we call  $(H_X, \Gamma, \varphi)$  a covariant system.

**Definition 4** ([30]). A covariant system  $(H_X, \Gamma, \varphi)$  is called admissible if

- (1) the  $\Gamma$ -action on  $X$  is proper and cocompact;
- (2)  $H_X$  is a nondegenerate standard  $X$ -module;
- (3) for each  $x \in X$ , the stabilizer group  $\Gamma_x$  acts on  $H_X$  regularly in the sense that the action is isomorphic to the action of  $\Gamma_x$  on  $l^2(\Gamma_x) \otimes H$  for some infinite dimensional Hilbert space  $H$ . Here  $\Gamma_x$  acts on  $l^2(\Gamma_x)$  by translations and acts on  $H$  trivially.

We remark that for each locally compact metric space  $X$  with a proper and cocompact isometric action of  $\Gamma$ , there exists an admissible covariant system  $(H_X, \Gamma, \varphi)$ . Also, we point out that the condition (3) above can be dropped if  $\Gamma$  acts freely on  $X$ . If no confusion arises, we will denote a covariant system  $(H_X, \Gamma, \varphi)$  by  $H_X$  and call it an admissible  $(X, \Gamma)$ -module.

*Remark 5.* For each  $(X, \Gamma)$  above, there always exists an admissible  $(X, \Gamma)$ -module  $\mathcal{H}$ . In particular,  $H \oplus \mathcal{H}$  is an admissible  $(X, \Gamma)$ -module for every  $(X, \Gamma)$ -module  $H$ .

**Definition 6.** Let  $X$  be a locally compact metric space  $X$  with a proper and cocompact isometric action of  $\Gamma$ . If  $H_X$  is an admissible  $(X, \Gamma)$ -module, then we define  $C^*(X)^\Gamma$  to be the  $C^*$ -algebra generated by all  $\Gamma$ -invariant locally compact operators with finite propagations in  $\mathcal{B}(H_X)$ .

Since the action of  $\Gamma$  on  $X$  is cocompact, it is known that in this case  $C^*(X)^\Gamma$  is  $*$ -isomorphic to  $C_r^*(\Gamma) \otimes \mathcal{K}$ , where  $C_r^*(\Gamma)$  is the reduced group  $C^*$ -algebra of  $\Gamma$  and  $\mathcal{K}$  is the algebra of all compact operators.

Similarly, we can also define  $D^*(X)^\Gamma$ ,  $C_L^*(X)^\Gamma$ ,  $D_L^*(X)^\Gamma$ ,  $C_{L,0}^*(X)^\Gamma$ ,  $D_{L,0}^*(X)^\Gamma$ ,  $C_L^*(Y; X)^\Gamma$  and  $C_{L,0}^*(Y; X)^\Gamma$ .

*Remark 7.*  $C^*(X) = C^*(X, H_X)$  does not depend on the choice of the standard nondegenerate  $X$ -module  $H_X$ . The same holds for  $D^*(X)$ ,  $C_L^*(X)$ ,  $D_L^*(X)$ ,  $C_{L,0}^*(X)$ ,  $D_{L,0}^*(X)$ ,  $C_L^*(Y; X)$ ,  $C_{L,0}^*(Y; X)$  and their  $\Gamma$ -invariant versions.

*Remark 8.* Note that we can also define maximal versions of all the  $C^*$ -algebras above.



## 2.3 Index map, local index map and higher rho invariant

In this subsection, we recall the constructions of the index map, the local index map (cf. [28, 30]) and the higher rho invariant.

Let  $X$  be a locally compact metric space with a proper and cocompact isometric action of  $\Gamma$ . We recall the definition of the  $K$ -homology groups  $K_j^\Gamma(X)$ ,  $j = 0, 1$ . They are generated by certain cycles modulo certain equivalence relations (cf. [16]):

- (i) an even cycle for  $K_0^\Gamma(X)$  is a pair  $(H_X, F)$ , where  $H_X$  is an admissible  $(X, \Gamma)$ -module and  $F \in \mathcal{B}(H_X)$  such that  $F$  is  $\Gamma$ -equivariant,  $F^*F - I$  and  $FF^* - I$  are locally compact and  $[F, f] = Ff - fF$  is compact for all  $f \in C_0(X)$ .
- (ii) an odd cycle for  $K_1^\Gamma(X)$  is a pair  $(H_X, F)$ , where  $H_X$  is an admissible  $(X, \Gamma)$ -module and  $F$  is a self-adjoint operator in  $\mathcal{B}(H_X)$  such that  $F^2 - I$  is locally compact and  $[F, f]$  is compact for all  $f \in C_0(X)$ .

*Remark 9.* In the general case where the action of  $\Gamma$  on  $X$  is not necessarily cocompact, we define

$$K_i^\Gamma(X) = \varinjlim_{Y \subseteq X} K_i^\Gamma(Y)$$

where  $Y$  runs through all closed  $\Gamma$ -invariant subsets of  $X$  such that  $Y/\Gamma$  is compact.

### 2.3.1 Index map and local index map

Now let  $(H_X, F)$  be an even cycle for  $K_0^\Gamma(X)$ . If  $H_X$  is not admissible, then we take the direct sum  $H_X$  with an admissible  $(X, \Gamma)$ -module  $\mathcal{H}$  and define  $F' = F \oplus 1$ . It is easy to see that  $(H_X \oplus \mathcal{H}, F')$  is equivalent to  $(H_X, F)$  in  $K_0^\Gamma(X)$ . So without loss of generality, we assume  $H_X$  is an admissible  $(X, \Gamma)$ -module.

Let  $\{U_i\}$  be a  $\Gamma$ -invariant locally finite open cover of  $X$  with  $\text{diameter}(U_i) < c$  for some fixed  $c > 0$ . Let  $\{\phi_i\}$  be a  $\Gamma$ -invariant continuous partition of unity subordinate to  $\{U_i\}$ . We define

$$G = \sum_i \phi_i^{1/2} F \phi_i^{1/2},$$

where the sum converges in strong topology. It is not difficult to see that  $(H_X, G)$  is equivalent to  $(H_X, F)$  in  $K_0^\Gamma(X)$ . By using the fact that  $G$  has finite propagation, we see that  $G$  is a multiplier of  $C^*(X)^\Gamma$  and  $G$  is a unitary modulo  $C^*(X)^\Gamma$ . Now by the standard construction in Section 2.1 above,  $G$  produces a class  $[G] \in K_0(C^*(X)^\Gamma)$ . We define the index of  $(H_X, F)$  to be  $[G]$ .

From now on, we denote this index class of  $(H_X, F)$  by  $\text{Ind}(H_X, F)$  or simply  $\text{Ind}(F)$  if no confusion arises.

For each  $n \in \mathbb{N}$ , let  $\{U_{n,j}\}$  be a  $\Gamma$ -invariant locally finite open cover of  $X$  with  $\text{diameter}(U_{n,j}) < 1/n$  and  $\{\phi_{n,j}\}$  be a  $\Gamma$ -invariant continuous partition of unity subordinate to  $\{U_{n,j}\}$ . We define

$$G(t) = \sum_j (1 - (t - n)) \phi_{n,j}^{1/2} G \phi_{n,j}^{1/2} + (t - n) \phi_{n+1,j}^{1/2} G \phi_{n+1,j}^{1/2}$$

for  $t \in [n, n+1]$ .



*Remark 10.* Here by convention, we assume that the open cover  $\{U_{0,j}\}$  is the trivial cover  $\{X\}$  when  $n = 0$ .

Then  $G(t), 0 \leq t < \infty$ , is a multiplier of  $C_L^*(X)^\Gamma$  and a unitary modulo  $C_L^*(X)^\Gamma$ , hence defines a class in  $K_0(C_L^*(X)^\Gamma)$ . We define the local index of  $(H_X, F)$  to be

$$[G(t)] \in K_0(C_L^*(X)^\Gamma).$$

From now on, we denote this local index class of  $(H_X, F)$  by  $\text{Ind}_L(H_X, F)$  or simply  $\text{Ind}_L(F)$  is no confusion arises.

Now let  $(H_X, F)$  be an odd cycle in  $K_1^\Gamma(X)$ . With the same notation from the above, we set  $q = \frac{G+1}{2}$ . Then the index class of  $(H_X, F)$  is defined to be  $[e^{2\pi i q}] \in K_1(C^*(X)^\Gamma)$ . For the local index map  $\text{Ind}_L : K_1^\Gamma(X) \rightarrow K_1(C_L^*(X)^\Gamma)$ , one simply uses  $q(t) = \frac{G(t)+1}{2}$  in place of  $q$ .

Now suppose  $M$  is an odd dimensional complete spin manifold without boundary and we fix a spin structure on  $M$ . Assume that there is a discrete group  $\Gamma$  acting on  $M$  properly and cocompactly by isometries. In addition, we assume the action of  $\Gamma$  preserves the spin structure on  $M$ . Let  $\mathcal{S}$  be the spinor bundle over  $M$  and  $D = D_M$  be the associated Dirac operator on  $M$ . Let  $H_M = L^2(M, \mathcal{S})$  and

$$F = D(D^2 + 1)^{-1/2}.$$

Then  $(H_M, F)$  defines a class in  $K_1^\Gamma(M)$ . By the above discussion, we have both the index class and the local index class of  $(H_M, F)$ . We shall denote them by  $\text{Ind}(D_M) \in K_1(C^*(M)^\Gamma)$  and  $\text{Ind}_L(D_M) \in K_1(C_L^*(M)^\Gamma)$  respectively.

The even dimensional case is essentially the same, where one needs to work with the natural  $\mathbb{Z}/2\mathbb{Z}$ -grading on the spinor bundle. We leave the details to the interested reader (see also Section 3.2 below).

*Remark 11.* If we use the maximal versions of the  $C^*$ -algebras, then the same construction defines an index class (resp. a local index class) in the  $K$ -theory of the maximal version of corresponding  $C^*$ -algebra.

### 2.3.2 Higher rho invariant

With  $M$  from above, suppose in addition  $M$  is endowed with a complete Riemannian metric  $h$  whose scalar curvature  $\kappa$  is positive everywhere, then the associated Dirac operator  $(D_M, h)$  naturally defines a class in  $K_1(C_{L,0}^*(M)^\Gamma)$ . Indeed, recall that

$$D^2 = \nabla^* \nabla + \frac{\kappa}{4},$$

where  $\nabla : C^\infty(M, \mathcal{S}) \rightarrow C^\infty(M, T^*M \otimes \mathcal{S})$  is the connection and  $\nabla^*$  is the adjoint of  $\nabla$ . It follows immediately that  $D(D^2 + 1)^{-1/2}$  is invertible in this case. In fact, in this case we can define

$$F = D|D|^{-1}.$$

Then  $\frac{F+1}{2}$  is a genuine projection. Let  $G$  be as above. Then  $q = \frac{G+1}{2}$  is a  $\delta$ -quasi-projection with  $\delta$  sufficiently small<sup>1</sup>, where  $q$  being a  $\delta$ -quasi-projection means that  $q^* = q$  and

$$\|q^2 - q\| < \delta.$$

Notice that  $e^{2\pi i q}$  is sufficiently close to 1, when  $q$  is a  $\delta$ -quasi-projection for  $\delta$  sufficiently small.

Now define a similar path of elements  $G(t), 0 \leq t < \infty$ , as above. Let  $q(t) = \frac{G(t)+1}{2}$ . We form the path  $e^{2\pi i q(t)}, 0 \leq t < \infty$ , in  $(C_{L,0}^*(M)^\Gamma)^+$ . Since  $e^{2\pi i q(0)} = e^{2\pi i q}$  is sufficiently close to 1, we can connect  $e^{2\pi i q(0)}$  to 1 by a path of invertible elements (e.g. the linear path between  $e^{2\pi i q(0)}$  and 1). Now define

$$u(t) = \begin{cases} te^{2\pi i q(0)} + (1-t) & \text{if } t \in [0, 1]; \\ e^{2\pi i q(t-1)} & \text{if } t \geq 1. \end{cases}$$

Then  $u(t), 0 \leq t < \infty$ , gives rise to a class in  $K_1(C_{L,0}^*(M)^\Gamma)$ . This class is called the higher rho invariant of  $(D_M, h)$  and will be denoted by  $\rho(D_M, h)$  from now on.

Again, the even dimensional case is essentially the same, where one needs to work with the natural  $\mathbb{Z}/2\mathbb{Z}$ -grading on the spinor bundle. We leave the details to the interested reader (see also Section 3.2 below).

*Remark 12.* If we use the maximal version of the  $C^*$ -algebra  $C_{L,0}^*(M)^\Gamma$ , then the same construction defines a higher rho invariant in the  $K$ -theory of the maximal version of  $C_{L,0}^*(M)^\Gamma$ .

### 3 Manifolds with positive scalar curvature on the boundary

In this section, we discuss about the higher index classes of Dirac operators on spin manifolds with boundary, where the boundary is endowed with a positive scalar curvature metric.

Throughout the section, let  $M$  be a complete spin manifold with boundary  $\partial M$  such that

- (i) the metric on  $M$  has product structure near  $\partial M$  and its restriction on  $\partial M$  has positive scalar curvature;
- (ii) there is an proper and cocompact isometric action of a discrete group  $\Gamma$  on  $M$ ;
- (iii) the action of  $\Gamma$  preserves the spin structure of  $M$ .

We attach an infinite cylinder  $\mathbb{R}_{\geq 0} \times \partial M = [0, \infty) \times \partial M$  to  $M$ . If we denote the Riemannian metric on  $\partial M$  by  $h$ , then we endow  $\mathbb{R}_{\geq 0} \times \partial M$  by the standard

---

<sup>1</sup>This can always be achieved by choose an appropriate partition of unity. Note that in this case, although  $G$  has finite propagation, the propagation may be very large. In other words, in general the smallness of the propagation of  $G$  and the invertibility of  $G$  cannot be achieved at the same time.

product metric  $dr^2 + h$ . Notice that all geometric structures on  $M$  extend naturally to  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$ . The action of  $\Gamma$  on  $M$  also extends to  $M_\infty$ .

Let us set  $M_n = M \cup_{\partial M} ([0, n] \times \partial M)$ , for  $n \geq 0$ . In particular,  $M_0 = M$ . Again, the action of  $\Gamma$  on  $M$  extends naturally to  $M_n$ , where  $\Gamma$  acts on  $[0, n]$  trivially.

*Remark 13.* In fact, without loss of generality, we assume that we have fixed an identification of a neighborhood of  $\partial M$  in  $M$  with  $[-3, 0] \times \partial M$ . For  $0 \leq k \leq 3$ , we will write

$$M_{-k} = M \setminus ((-k, 0] \times \partial M).$$

This will be used later for notational simplification. See Figure 1 below.

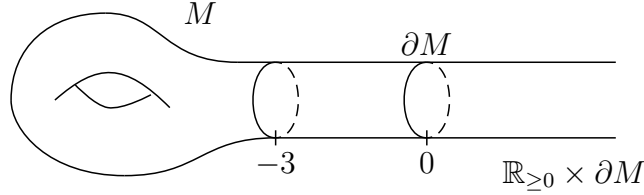


Figure 1: Attaching  $\mathbb{R}_{\geq 0} \times \partial M$  to  $M$ .

Now let  $D = D_{M_\infty}$  be the associated Dirac operator on  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$ .

**Claim.** *In the above setting,  $D$  in fact defines an index class, denoted by  $\text{Ind}(D)$ , in  $K_i(C^*(M_n)^\Gamma) \cong K_i(C^*(M)^\Gamma)$ .*

In this section, we carry out a detailed construction to verify this claim.

*Remark 14.* The action of  $\Gamma$  on  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$  is not cocompact, that is,  $M_\infty/\Gamma$  is not compact.

*Remark 15.* The above claim is not true in general, if we drop the positive scalar curvature assumption near the boundary,.

Let  $\kappa$  be the scalar curvature function on  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$  and  $\rho$  a  $\Gamma$ -invariant nonnegative smooth function on  $M_\infty$  satisfying:

- (a)  $\text{Supp}(\rho) \subseteq M_n = M \cup_{\partial M} ([0, n] \times \partial M)$  for some  $n \in \mathbb{N}$ ;
- (b) there exists  $c > 0$  such that  $\rho(y) + \frac{\kappa(y)}{4} > c$  for all  $y \in M_\infty$ .

Define

$$F = F_\rho = \frac{D}{\sqrt{D^2 + \rho}}.$$

Recall that

$$\frac{1}{\sqrt{x}} = \frac{2}{\pi} \int_0^\infty \frac{1}{x + \lambda^2} d\lambda.$$

It follows that

$$\frac{1}{\sqrt{D^2 + \rho}} = \frac{2}{\pi} \int_0^\infty (D^2 + \rho + \lambda^2)^{-1} d\lambda.$$

Now by using the equality

$$(D^2 + \rho + \lambda^2)^{-1}D - D(D^2 + \rho + \lambda^2)^{-1} = (D^2 + \rho + \lambda^2)^{-1}[D, \rho](D^2 + \rho + \lambda^2)^{-1},$$

we have

$$\begin{aligned} F^2 &= D(D^2 + \rho)^{-1/2}D(D^2 + \rho)^{-1/2} \\ &= D\left(\frac{2}{\pi}\int_0^\infty (D^2 + \rho + \lambda^2)^{-1}d\lambda\right)D(D^2 + \rho)^{-1/2} \\ &= \left(D^2 \cdot \frac{2}{\pi}\int_0^\infty (D^2 + \rho + \lambda^2)^{-1}d\lambda + D \cdot \frac{2}{\pi}\int_0^\infty R(\lambda)d\lambda\right)(D^2 + \rho)^{-1/2} \\ &= D^2(D^2 + \rho)^{-1} + DR(D^2 + \rho)^{-1/2} \\ &= 1 - \rho(D^2 + \rho)^{-1} + DR(D^2 + \rho)^{-1/2} \end{aligned}$$

where  $R = \frac{2}{\pi}\int_0^\infty R(\lambda)d\lambda$  with

$$R(\lambda) = (D^2 + \rho + \lambda^2)^{-1}[D, \rho](D^2 + \rho + \lambda^2)^{-1}.$$

Since  $\text{Supp}(\rho)$  and  $\text{Supp}([D, \rho])$  are  $\Gamma$ -cocompact, it follows that both  $\rho(D^2 + \rho)^{-1}$  and  $DR(D^2 + \rho)^{-1/2}$  are in  $C^*(M_\infty)^\Gamma$ .

In fact, for  $\forall \varepsilon > 0$ , there exists such a  $\rho$  satisfying:

- (i)  $\rho \equiv C$  on  $M_{n_1} = M \cup_{\partial M} ([0, n_1] \times \partial M)$  and  $\rho \equiv 0$  on  $[n_2, \infty) \times \partial M$  for some  $n_2 > n_1 \geq 0$  and some constant  $C > 0$ ;
- (ii)  $\rho(x, y) = \rho(x, y')$  for all  $y, y' \in \partial M$ , where  $(x, y) \in (n_1, n_2) \times \partial M$ . In other words,  $\rho$  is constant along  $\partial M$ . In particular, it follows that

$$[D, \rho] = \rho'$$

$$[D^2, \rho] = \rho'' + 2\rho'D$$

where  $\rho' = \partial_x \rho$  and  $\rho'' = \partial_x^2 \rho$ ;

- (iii)  $\|(D^2 + \rho)^{-1}[D^2, \rho](D^2 + \rho)^{-1}\| < \varepsilon$  and  $\|DR(D^2 + \rho)^{-1/2}\| < \varepsilon$ .

Let  $a_0$  be a real number such that  $\text{Supp}(\rho) \subset M_{a_0}$ . Let us decompose the space  $L^2(M_\infty, \mathcal{S})$  into a direct sum  $L^2(M_a, \mathcal{S}) \oplus L^2(\mathbb{R}_{\geq a} \times \partial M, \mathcal{S})$  for  $a \geq a_0$ . If we write

$$F^2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with respect to this decomposition, then it follows that

$$\|A_{12}\| < 2\varepsilon, \|A_{21}\| < \varepsilon \text{ and } \|A_{22} - 1\| < \varepsilon.$$

Here to see the estimate for the term  $A_{12}$ , we can rewrite

$$F^2 = 1 - (D^2 + \rho)^{-1}\rho - (D^2 + \rho)^{-1}[D^2, \rho](D^2 + \rho)^{-1} + DR(D^2 + \rho)^{-1/2},$$

since

$$\rho(D^2 + \rho)^{-1} = (D^2 + \rho)^{-1}\rho + (D^2 + \rho)^{-1}[D^2, \rho](D^2 + \rho)^{-1}.$$

### 3.1 Invertibles

Now suppose the dimension of  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$  is odd. Let  $F = F_\rho$  as above and  $P = \frac{F+1}{2}$ . Notice that

$$P^2 - P = \frac{F^2 - 1}{4} = \frac{1}{4} \begin{pmatrix} A_{11} - 1 & A_{12} \\ A_{21} & A_{22} - 1 \end{pmatrix}.$$

Now for  $\forall \delta > 0$  and  $\forall n \in \mathbb{N}$ , we can choose a  $\Gamma$ -invariant locally finite open cover  $\{U_{n,j}\}$  of  $M_\infty$  and a set of  $\Gamma$ -invariant partition functions  $\{\varphi_{n,j}\}$  subordinate to  $\{U_{n,j}\}$  such that

- (a) if  $U_{n,j} \subset M$ , then  $\text{diameter}(U_{n,j}) < 1/n$ , where we call  $1/n$  the propagation upper bound on the subset  $M \subset M_\infty$ ;
- (b) if  $U_{n,j}$  is not contained in  $M$ , then  $\|[F, \varphi_j^{1/2}]\| < \delta$ ;
- (c) if  $U_{n,j} \subset \mathbb{R}_{\geq 0} \times \partial M$ , then  $U_{n,j}$  intersects at most two open sets  $U_{n,j'}$  with  $j' \neq j$ .

*Remark 16.* When  $n = 0$ , we assume that the open cover  $\{U_{0,j}\}$  is the trivial cover which consists of a single set  $M_\infty$ .

Let us define

$$F(t) = \sum_j (1 - (t - n)) \varphi_{n,j}^{1/2} F \varphi_{n,j}^{1/2} + (t - n) \varphi_{n+1,j}^{1/2} F \varphi_{n+1,j}^{1/2} \quad (1)$$

for  $t \in [n, n+1]$ . It follows immediately that

- (i) For each fixed  $t \in [0, \infty)$ ,  $F(t)$  defines the same class  $\text{Ind}(D_{M_\infty}) \in K_1(C^*(M_\infty)^\Gamma)$ ;
- (ii) the propagation  $F(t)$  restricted on  $M_{-1} = M \setminus ((-1, 0] \times \partial M)$  goes to 0, as  $t \rightarrow \infty$ .

Now let us write

$$F(t)^2 = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$$

with respect to the decomposition  $L^2(M_a, \mathcal{S}) \oplus L^2(\mathbb{R}_{\geq a} \times \partial M, \mathcal{S})$  as before. Notice that there exists  $\Lambda_0$  such that

$$\|A_{12} - B_{12}(t)\| < 2\delta, \|A_{21} - B_{21}(t)\| < 2\delta \quad \text{and} \quad \|A_{22} - B_{22}(t)\| < 2\delta$$

for the decomposition  $L^2(M_a, \mathcal{S}) \oplus L^2(\mathbb{R}_{\geq a} \times \partial M, \mathcal{S})$  with  $a > \Lambda_0$  and all  $t \in [0, \infty)$ .

Now let  $P(t) = \frac{F(t)+1}{2}$ , then

$$P(t)^2 - P(t) = \frac{F(t)^2 - 1}{4} = \frac{1}{4} \begin{pmatrix} B_{11}(t) - 1 & B_{12}(t) \\ B_{21}(t) & B_{22}(t) - 1 \end{pmatrix}.$$

It follows that  $\|E(t) - (P(t)^2 - P(t))\| < (\varepsilon + \delta)/2$ , where

$$E(t) = \frac{1}{4} \begin{pmatrix} B_{11}(t) - 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Recall that  $e^{2\pi i P(t)}$  is a representative of the index class  $\text{Ind}(D_{M_\infty}) \in K_1(C^*(M_\infty)^\Gamma)$ , for each fixed  $t \in [0, \infty)$ . Now for each  $N \in \mathbb{N}$ , consider the polynomial

$$f_N(x) = \sum_{n=0}^N \frac{(2\pi i)^n}{n!} x^n.$$

Write  $Q = P(t)$ . Since  $Q$  has bounded spectrum, we can choose  $N$  such that

$$\|f_N(Q) - e^{2\pi i Q}\|$$

is as small as we want. Note that for all  $n \geq 2$ , we have

$$Q^n - Q = \left( \sum_{j=0}^{n-2} Q^j \right) (Q^2 - Q).$$

It follows that

$$\begin{aligned} f_N(Q) &= \sum_{n=0}^N \frac{(2\pi i)^n}{n!} Q^n \\ &= 1 + \left( \sum_{n=1}^N \frac{(2\pi i)^n}{n!} \right) Q + \sum_{n=1}^N \frac{(2\pi i)^n}{n!} (Q^n - Q) \\ &= 1 + \left( \sum_{n=1}^N \frac{(2\pi i)^n}{n!} \right) Q + \left( \sum_{n=1}^N \sum_{j=0}^{n-2} \frac{(2\pi i)^n}{n!} Q^j \right) (Q^2 - Q) \end{aligned}$$

**Definition 17.** We define

$$u(t) = 1 + \left( \sum_{n=1}^N \sum_{j=1}^{n-2} \frac{(2\pi i)^n}{n!} P(t)^j \right) E(t).$$

Now we recall some basic estimates. First, it is obvious that

$$\sum_{n=1}^N \frac{(2\pi i)^n}{n!} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Moreover, if we denote by  $\zeta(t) = \max\{\|P(t)\|, 1\}$ , then

$$\left\| \sum_{n=1}^N \sum_{j=1}^{n-2} \frac{(2\pi i)^n}{n!} P(t)^j \right\| \leq 2\pi \zeta(t) \cdot e^{2\pi \zeta(t)},$$

for all  $N \in \mathbb{N}$ .

It follows immediately that

$$\|u(t) - f_N(P(t))\| < 2\pi \zeta(t) \cdot e^{2\pi \zeta(t)} (\varepsilon + \delta)$$

for sufficiently large  $N$ .

Notice that  $\|P(t)\|$  may change as  $t$  varies. An essential property of  $F(t)$  we constructed above is that the propagation upper bound on  $M_{-1} = M \setminus ((-1, 0] \times \partial M)$  goes to 0 as  $t \rightarrow \infty$ . However, we can choose  $\varepsilon$  and  $\delta$  to be arbitrarily small, independent of the upper bound of  $F(t)$  on  $M_2 = M \cup_{\partial M} ([0, 2] \times \partial M)$ . It follows that, for each  $t$ , we can choose  $\varepsilon = \varepsilon(t)$  and  $\delta = \delta(t)$  to be so small<sup>2</sup> that  $u(t)$  is sufficiently close to  $e^{2\pi i P(t)}$  and in particular is invertible. Let  $\lambda(t)$  be the propagation of  $P(t)$  and<sup>3</sup>  $\Lambda(t) = \Lambda_0 + N \cdot \lambda(t)$ . Recall that

$$E(t) = \begin{pmatrix} B_{11}(t) - 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition  $L^2(M_{\Lambda_0}, \mathcal{S}) \oplus L^2(\mathbb{R}_{\geq \Lambda_0} \times \partial M, \mathcal{S})$ . It follows immediately that  $u(t)$  preserves the subspace  $L^2(M_{\Lambda(t)}, \mathcal{S})$ . Moreover,  $u(t) = 1$  on  $L^2(\mathbb{R}_{\geq \Lambda(t)} \times \partial M, \mathcal{S})$ .

Now we need to justify that, for each  $t \in [0, \infty)$ , the element  $u(t)$  defines a class in  $K_1(C^*(M_{\Lambda(t)})^\Gamma)$ . After all,  $u(t) \notin (C^*(M_{\Lambda(t)})^\Gamma)^+$  in general.

**Definition 18** ( $\tau$ -closeness in the odd case). Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\mathcal{J}$  a closed ideal of  $\mathcal{A}$ . For  $\tau > 0$ , we call an invertible element  $x \in \mathcal{A}$  is  $\tau$ -close to  $\mathcal{J}$  if there exists an element  $y \in \mathcal{J}^+$  such that

$$\|x - y\| < \min \left\{ \tau, \frac{\tau}{\|x^{-1}\|} \right\}.$$

Recall that  $f_N(P(t)) \in (C^*(M_\infty)^\Gamma)^+$ . Moreover, it follows from our construction above that, for  $\forall \tau > 0$ , we can choose  $u(t)$ , which is an element in  $D^*(M_{\Lambda(t)})^\Gamma$ , to be  $\tau$ -close to  $(C^*(M_{\Lambda(t)})^\Gamma)^+$ . In fact, for sufficiently small<sup>4</sup>  $\tau > 0$ , we can choose  $u_1(t) \in (C^*(M_{\Lambda(t)})^\Gamma)^+$  such that  $\|u_1(t) - u(t)\| < \tau$  and  $u_1(t) = u(t)$  on  $L_2(M_1, \mathcal{S})$ . In particular,  $u_1(t)$  is invertible.

**Definition 19.** With the same notation as above, for each fixed  $t_0 \in [0, \infty)$ , the class associated to  $u(t_0)$  in  $K_1(C^*(M_{\Lambda(t)})^\Gamma)$  is defined to be  $[u_1(t_0)]$ .

It is easy to see that this class is independent of the choice of  $u_1(t_0)$ . So we will simply denote it by  $[u(t_0)]$  from now on. Moreover,  $M_{\Lambda(t_0)}$  is clearly ( $\Gamma$ -equivariantly) coarsely equivalent to  $M_0 = M$ . So we can think of  $\text{Ind}(D_{M_\infty}) = [u(t_0)]$  as a class in  $K_1(C^*(M)^\Gamma)$ .

## 3.2 Idempotents

The even dimensional case is parallel to the odd dimensional case above. We will briefly go through the construction but leave out the details.

---

<sup>2</sup>More precisely, this can be achieved by choosing appropriately the function  $\rho$  and those of  $\{\varphi_{n,j}\}$  that have support in  $\mathbb{R}_{\geq 2} \times \partial M$ .

<sup>3</sup>To be more precise,  $\Lambda_0 = \Lambda_0(t)$  and  $N = N(t)$  also depend on  $t$ .

<sup>4</sup>For example, any positive number  $\tau < \frac{1}{10}$  suffices.



Now we assume that  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$  is even dimensional. Then the associated Dirac operator  $D_{M_\infty}$  is an odd operator with respect to the natural  $\mathbb{Z}/2\mathbb{Z}$ -grading on the spinor bundle  $\mathcal{S}$ . Let  $F(t)$  be as in formula (1) above. This time,  $F(t)$  is also odd-graded and let us write

$$F(t) = \begin{pmatrix} 0 & U(t) \\ V(t) & 0 \end{pmatrix}$$

with respect to the  $\mathbb{Z}/2\mathbb{Z}$ -grading. Notice that

$$F(t)^2 = \begin{pmatrix} U(t)V(t) & 0 \\ 0 & V(t)U(t) \end{pmatrix}.$$

Now with respect to a decomposition  $L^2(M_a, \mathcal{S}) \oplus L^2(\mathbb{R}_{\geq a} \times \partial M, \mathcal{S})$  of the Hilbert space  $L^2(M_\infty, \mathcal{S})$ , let us write

$$U(t)V(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \quad \text{and} \quad V(t)U(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix}.$$

Then, without loss of generality, we can assume that the operator norms of  $B_{12}(t)$ ,  $B_{21}(t)$ ,  $(B_{22}(t) - 1)$ ,  $C_{12}(t)$ ,  $C_{21}(t)$  and  $(C_{22}(t) - 1)$  are all sufficiently small. If we write

$$W(t) = \begin{pmatrix} 1 & U(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & U(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then for each fixed  $t \in [0, \infty)$ , the index class of  $D_{M_\infty}$  is

$$\text{Ind}(D_{M_\infty}) = [p(t)] - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} p(t) &= W(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W(t)^{-1} \\ &= \begin{pmatrix} U(t)V(t) + U(t)V(t)(1 - U(t)V(t)) & (2 + U(t)V(t))U(t)(1 - V(t)U(t)) \\ V(t)(1 - U(t)V(t)) & (1 - U(t)V(t))^2 \end{pmatrix}. \end{aligned}$$

Now if we set

$$Z_1(t) = \begin{pmatrix} 1 - B_{11}(t) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_2(t) = \begin{pmatrix} 1 - C_{11}(t) & 0 \\ 0 & 0 \end{pmatrix},$$

then  $\|Z_1(t) - (1 - U(t)V(t))\|$  and  $\|Z_2(t) - (1 - V(t)U(t))\|$  are sufficiently small. Define

$$q(t) = \begin{pmatrix} 1 - Z_1(t)^2 & (2 + U(t)V(t))U(t)Z_2(t) \\ V(t)Z_1(t) & Z_1(t)^2 \end{pmatrix}.$$

Similar to the odd dimensional case, it is not difficult to see that there exists  $\Lambda(t) > 0$  such that  $q(t)$  preserves the subspace  $L^2(M_{\Lambda(t)}, \mathcal{S})$  and  $q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  on  $L^2(\mathbb{R}_{\geq \Lambda(t)} \times \partial M, \mathcal{S})$ . Furthermore, we have the propagation of  $q(t)$  on  $M_{-1} = M \setminus ((-1, 0] \times \partial M)$  go to zero as  $t \rightarrow \infty$ . Again,  $q(t) \notin (C^*(M_{\Lambda(t)})^\Gamma)^+$  in general. To justify that the element  $q(t)$  in fact defines a class in  $K_0(C^*(M_{\Lambda(t)})^\Gamma)$  for each  $t \in [0, \infty)$ , we introduce the following definition.

**Definition 20** ( $\tau$ -closeness in the even case). Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\mathcal{J}$  a closed ideal of  $\mathcal{A}$ . For  $\tau > 0$ , we call an element  $x \in \mathcal{A}$  is  $\tau$ -close to  $\mathcal{J}$  if there exists an element  $y \in \mathcal{J}^+$  such that

$$\|x - y\| < \min \left\{ \tau, \frac{\tau}{\|x\|} \right\}.$$

Now for each  $t_0 \in [0, \infty)$ , we choose an element  $q_1(t_0)$  in  $(C^*(M_{\Lambda(t_0)})^\Gamma)^+$  that is  $\tau$ -close to  $q(t_0)$  with  $\tau$  sufficiently small. It is not difficult to see that  $q_1(t_0)$  is almost an idempotent and defines a class in  $K_0(C_*(M_{\Lambda(t_0)})^\Gamma)$  (cf. [27, Section 2.2]). Moreover, this class in  $K_0(C_*(M_{\Lambda(t_0)})^\Gamma)$  is independent of the choice of such  $q_1(t_0)$ , hence will be denoted by  $[q(t_0)]$  from now on. Again, we can think of  $\text{Ind}(D_{M_\infty}) = [q(t_0)]$  as a class in  $K_0(C^*(M)^\Gamma) \cong K_0(C^*(M_{\Lambda(t_0)})^\Gamma)$ .

*Remark 21.* The same construction works if we switch to the maximal versions of all the  $C^*$ -algebras above.

## 4 Main theorem

In this section, we prove the main theorem of this paper.

Let  $M$  be a complete spin manifold with boundary  $\partial M$  such that

- (i) the metric on  $M$  has product structure near  $\partial M$  and its restriction on  $\partial M$ , denoted by  $h$ , has positive scalar curvature;
- (ii) there is an proper and cocompact isometric action of a discrete group  $\Gamma$  on  $M$ ;
- (iii) the action of  $\Gamma$  preserves the spin structure of  $M$ .

Recall that we denote by  $M_\infty = M \cup_{\partial M} (\mathbb{R}_{\geq 0} \times \partial M)$  and  $M_n = M \cup_{\partial M} ([0, n] \times \partial M)$  for  $n \geq 0$  (cf. Section 3). We denote the associated Dirac operator on  $M_\infty$  by  $D_{M_\infty}$  and the associated Dirac operator on  $\partial M$  by  $D_{\partial M}$ . Let  $m = \dim M$ . Recall the following facts:

- (a) by our discussion in Section 3, the operator  $D_{M_\infty}$  actually defines an index class  $\text{Ind}(D_{M_\infty}) \in K_m(C^*(M)^\Gamma)$ ;
- (b) by the discussion in Section 2.3.2, there is a higher rho invariant  $\rho(D_{\partial M}, h) \in K_{m-1}(C_{L,0}(\partial M)^\Gamma)$  naturally associated to  $D_{\partial M}$ ;
- (c) the short exact sequence

$$0 \rightarrow C_{L,0}^*(M)^\Gamma \rightarrow C_L^*(M)^\Gamma \rightarrow C^*(M)^\Gamma \rightarrow 0$$

induces the following long exact sequence

$$\cdots \rightarrow K_i(C_L^*(M)^\Gamma) \rightarrow K_i(C^*(M)^\Gamma) \xrightarrow{\partial_i} K_{i-1}(C_{L,0}^*(M)^\Gamma) \rightarrow K_{i-1}(C_L^*(M)^\Gamma) \rightarrow \cdots;$$

- (d) the inclusion map  $\iota : C_{L,0}^*(\partial M; M)^\Gamma \rightarrow C_{L,0}^*(M)^\Gamma$  induces a natural map  $\iota_* : K_i(C_{L,0}^*(\partial M; M)^\Gamma) \rightarrow K_i(C_{L,0}^*(M)^\Gamma)$  (cf. Section 2.2);
- (e) there are natural isomorphisms (cf. [28])

$$\begin{array}{ccc}
& & K_i(C_{L,0}^*(\partial M; M)^\Gamma) \\
& \nearrow \cong & \uparrow \cong \\
K_i(C_{L,0}^*(\partial M)^\Gamma) & \xrightarrow{\cong} & K_i(C_{L,0}^*(\partial M; \mathbb{R}_{\leq 0} \times \partial M)^\Gamma)
\end{array}$$

where  $\partial M = \{0\} \times \partial M$  as a subset of  $\mathbb{R}_{\leq 0} \times \partial M$ .

We have the following main theorem of the paper. This extends a theorem of Piazza and Schick [21, Theorem 1.17] to all dimensions.

**Theorem 22.** *With the same notation as above, the image of  $\text{Ind}(D_{M_\infty})$  under the map*

$$K_m(C^*(M)^\Gamma) \xrightarrow{\partial_m} K_{m-1}(C_{L,0}^*(M)^\Gamma)$$

*lies in  $\iota_*(K_{m-1}(C_{L,0}^*(\partial M; M)^\Gamma))$ . Moreover,*

$$\partial_m(\text{Ind}(D_{M_\infty})) = \rho(D_{\partial M}, h)$$

*after the natural identification  $K_{m-1}(C_{L,0}^*(\partial M)^\Gamma) \cong K_{m-1}(C_{L,0}^*(\partial M; M)^\Gamma)$ .*

*Proof.* We will carry out a detailed proof for the odd dimensional case (i.e. when  $m$  is odd). The proof for the even dimensional case is essentially the same.

By the discussion in Section 3, we have

$$\text{Ind}(D_{M_\infty}) \in K_m(C^*(M)^\Gamma).$$

Moreover we can choose a uniformly continuous path of invertible elements<sup>5</sup>  $u(t) \in (C^*(M)^\Gamma)^+$  such that

- (i)  $[u(t)] = \text{Ind}(D_{M_\infty})$  for each fixed  $t \in [0, \infty)$ ,
- (ii) the propagation of  $u(t)$  restricted on  $L_2(M_{-1}, \mathcal{S})$  goes to 0, as  $t \rightarrow \infty$ ,

where we recall that  $M_{-1} = M \setminus ((-1, 0] \times \partial M)$ .

For each  $t \in [0, \infty)$ , choose  $\Gamma$ -invariant partition functions  $\{\varphi_t, \psi_t\}$  on  $M$  such that

- (a)  $\varphi_t(x) + \psi_t(x) = 1$ ;
- (b)  $\varphi_t(x) \equiv 1$  on  $M_{-2}$  and  $\varphi_t(x) \equiv 0$  on  $[-1, 0] \times \partial M$ ;
- (c)  $\psi_t \rightarrow \chi_{[-2, 0]}$ , as  $t \rightarrow \infty$ , where  $\chi_{[-2, 0]}$  is the characteristic function on  $[-2, 0] \times \partial M$ .

---

<sup>5</sup>Here we have identified  $C^*(M_{\Lambda(t)})^\Gamma$  with  $C^*(M)^\Gamma$ .

Now let  $\theta(t) \in C^\infty(-\infty, \infty)$  be a decreasing function such that  $\theta|_{(-\infty, 1]} = 1$  and  $\theta|_{[2, \infty)} = 0$ . We define

$$U(t) = \begin{cases} (1-t)u(0) + t \left( \varphi_0^{1/2} u(0) \varphi_0^{1/2} + \psi_0^{1/2} u(0) \psi_0^{1/2} \right) & \text{if } t \in [0, 1] \\ \varphi_{t-1}^{1/2} u(t-1) \varphi_{t-1}^{1/2} + \theta(t) \psi_{t-1}^{1/2} u(t-1) \psi_{t-1}^{1/2} & \text{if } t \in [1, \infty) \end{cases} \quad (2)$$

Notice that  $U(t) \in (C^*(M)^\Gamma)^+$  for each  $t \in [0, \infty)$ . By construction, the path  $U(t), 0 \leq t \leq \infty$ , is a lift of  $u(0) \in (C^*(M)^\Gamma)^+$  for the short exact sequence

$$0 \rightarrow C_{L,0}^*(M)^\Gamma \rightarrow C_L^*(M)^\Gamma \rightarrow C^*(M)^\Gamma \rightarrow 0.$$

We also apply the same argument to  $v(t) = u(t)^{-1}$  and define similarly

$$V(t), 0 \leq t < \infty, \text{ in } (C_L^*(M)^\Gamma)^+.$$

Now for each  $0 \leq t < \infty$ , we define an idempotent

$$p(t) = \begin{pmatrix} U(t)V(t) + U(t)V(t)(1 - U(t)V(t)) & (2 + U(t)V(t))U(t)(1 - V(t)U(t)) \\ V(t)(1 - U(t)V(t)) & (1 - U(t)V(t))^2 \end{pmatrix}.$$

By definition, we have

$$\partial_0(\text{Ind}(D_{M_\infty})) = [p(t)] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(C_{L,0}^*(M)^\Gamma).$$

By construction,  $p(t)$  preserves the decomposition

$$L^2(M_{-2}, \mathcal{S}) \oplus L^2([-2, 0] \times \partial M, \mathcal{S})$$

and  $p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  on  $L^2(M_{-2}, \mathcal{S})$ , where  $M_{-2} = M \setminus ((-2, 0] \times \partial M)$ . So we have

$$[p(t)] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(C_{L,0}^*([-2, 0] \times \partial M)^\Gamma) \cong K_0(C_{L,0}^*(\partial M)^\Gamma) \cong K_0(C_{L,0}^*(\partial M; M)^\Gamma). \quad (3)$$

Let us write

$$\mathcal{A}_+ = C_{L,0}^*(\mathbb{R}_{\geq 0} \times \partial M)^\Gamma, \quad \mathcal{A}_- = C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M)^\Gamma,$$

and

$$\mathcal{J}_- = C_{L,0}^*(\partial M; \mathbb{R}_{\leq 0} \times \partial M)^\Gamma,$$

where  $\partial M = \{0\} \times \partial M$  as a subset of  $\mathbb{R}_{\leq 0} \times \partial M$ . To prove the theorem, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} D_{M_\infty} \vdash \text{-----} \rightarrow (D_{\partial M}, h) & & \\ \downarrow \text{Ind} & & \downarrow \rho \\ K_1(C^*(M)^\Gamma) & & K_0(C_{L,0}^*(\partial M)^\Gamma) \\ \downarrow \partial & & \parallel \\ K_0(C_{L,0}^*(M)^\Gamma) & \longleftarrow & K_0(C_{L,0}^*(\partial M; M)^\Gamma) \cong K_0(\mathcal{J}_-) \end{array}$$

Notice that the following  $C^*$ -algebras are naturally isomorphic to each other:

$$\begin{aligned}\mathcal{A}_-/\mathcal{J}_- &\cong C_{L,0}^*(\mathbb{R} \times \partial M)^\Gamma / C_{L,0}^*(\mathbb{R}_{\geq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma \\ &\cong C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma / C_{L,0}^*(\partial M; \mathbb{R} \times \partial M)^\Gamma.\end{aligned}$$

Consider the following short exact sequence of  $C^*$ -algebras

$$0 \rightarrow C_{L,0}^*(\partial M; \mathbb{R} \times \partial M)^\Gamma \rightarrow \begin{array}{c} C_{L,0}^*(\mathbb{R}_{\geq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma \\ \oplus \\ C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma \end{array} \rightarrow C_{L,0}^*(\mathbb{R} \times \partial M)^\Gamma \rightarrow 0,$$

where  $\partial M = \{0\} \times \partial M$  as a subset of  $\mathbb{R} \times \partial M$ . This is a Mayer-Vietoris sequence for the decomposition  $\mathbb{R} \times \partial M = (\mathbb{R}_{\geq 0} \times \partial M) \cup_{(\{0\} \times \partial M)} (\mathbb{R}_{\leq 0} \times \partial M)$ . It is clear that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{L,0}^*(\partial M; \mathbb{R} \times \partial M)^\Gamma & \longrightarrow & \begin{array}{c} C_{L,0}^*(\mathbb{R}_{\geq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma \\ \oplus \\ C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma \end{array} & \longrightarrow & C_{L,0}^*(\mathbb{R} \times \partial M)^\Gamma \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{L,0}^*(\partial M; \mathbb{R} \times \partial M)^\Gamma & \longrightarrow & C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma & \longrightarrow & \mathcal{A}_-/\mathcal{J}_- \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathcal{J}_- & \longrightarrow & \mathcal{A}_- & \longrightarrow & \mathcal{A}_-/\mathcal{J}_- \longrightarrow 0 \end{array}$$

where the maps are all defined the obvious way. Recall that (cf. [28])

$$K_i(\mathcal{A}_+) = K_i(C_{L,0}^*(\mathbb{R}_{\geq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma),$$

$$K_i(\mathcal{A}_-) = K_i(C_{L,0}^*(\mathbb{R}_{\leq 0} \times \partial M; \mathbb{R} \times \partial M)^\Gamma),$$

and

$$K_i(\mathcal{J}_-) = K_i(C_{L,0}^*(\partial M)^\Gamma) = K_i(C_{L,0}^*(\partial M; \mathbb{R} \times \partial M)^\Gamma).$$

In particular, the second row and the third row of the above commutative diagram give rise to identical long exact sequence at  $K$ -theory level. Therefore, we have following commutative diagram:

$$\begin{array}{ccccc} & & (D_{\mathbb{R} \times \partial M}, dx^2 + h) & \dashrightarrow & (D_{\partial M}, h) \\ & & \downarrow \rho & & \downarrow \rho \\ K_1(\mathcal{A}_-) \oplus K_1(\mathcal{A}_+) & \longrightarrow & K_1(C_{L,0}^*(\mathbb{R} \times \partial M)^\Gamma) & \xrightarrow{\partial_{MV}} & K_0(C_{L,0}^*(\partial M)^\Gamma) \\ \downarrow \Theta_1 & & \downarrow \Theta_2 & & \downarrow \Theta_3 \\ K_1(\mathcal{A}_-) & \longrightarrow & K_1(\mathcal{A}_-/\mathcal{J}_-) & \xrightarrow{\delta} & K_0(\mathcal{J}_-) \\ & & & & \uparrow \partial_1 \\ & & & & K_1(C^*(M)^\Gamma) \\ & & & & \uparrow \text{Ind} \\ & & & & (D_{M_\infty}) \end{array}$$

where we have

- (i)  $\partial_{MV} [\rho(D_{\mathbb{R} \times \partial M}, dx^2 + h)] = \rho(D_{\partial M}, h)$  follows essentially from the Bott periodicity. Indeed,  $\rho(D_{\mathbb{R} \times \partial M}, dx^2 + h)$  in fact naturally lies in  $K_1(C_L^*(\mathbb{R}) \otimes C_{L,0}^*(\partial M)^\Gamma)$ . Consider

$$\begin{array}{ccccc}
(D_{\mathbb{R} \times \partial M}, dx^2 + h) & \dashrightarrow & (D_{\partial M}, h) \\
\downarrow \rho & & \downarrow \rho \\
K_1(C_L^*(\mathbb{R}_{\leq 0}) \otimes C_{L,0}^*(\partial M)^\Gamma) & \longrightarrow & K_1(C_L^*(\mathbb{R}) \otimes C_{L,0}^*(\partial M)^\Gamma) & \xrightarrow{\partial_{MV}} & K_0(C_L^*(\{0\}) \otimes C_{L,0}^*(\partial M)^\Gamma) \\
\oplus \\
K_1(C_L^*(\mathbb{R}_{\geq 0}) \otimes C_{L,0}^*(\partial M)^\Gamma) & \longrightarrow & K_1(C_L^*(\mathbb{R}) \otimes C_{L,0}^*(\partial M)^\Gamma) & \xrightarrow{\partial_{MV}} & K_0(C_L^*(\{0\}) \otimes C_{L,0}^*(\partial M)^\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(\mathcal{A}_-) \oplus K_1(\mathcal{A}_+) & \longrightarrow & K_1(C_{L,0}^*(\mathbb{R} \times \partial M)^\Gamma) & \xrightarrow{\partial_{MV}} & K_0(C_{L,0}^*(\partial M)^\Gamma)
\end{array}$$

then it suffices to verify that the following diagram commutes:

$$\begin{array}{ccc}
& D_{\mathbb{R}} \dashrightarrow 1 \\
& \downarrow \\
K_1(C_L^*(\mathbb{R}_{\leq 0})) \oplus K_1(C_L^*(\mathbb{R}_{\geq 0})) & \longrightarrow & K_1(C_L^*(\mathbb{R})) \xrightarrow{\partial_{MV}} K_0(C_L^*(\{0\}))
\end{array}$$

This follows from the Bott periodicity.

- (ii)  $\delta \circ \Theta_2 [\rho(D_{\mathbb{R} \times \partial M}, dx^2 + h)] = \partial_1(\text{Ind}(D_{M_\infty}))$  follows from our construction of  $p(t)$  in (3). In fact, if we apply the same construction of  $\partial_1(\text{Ind}(D_{M_\infty}))$  to define the class  $\delta \circ \Theta_2 [\rho(D_{\mathbb{R} \times \partial M}, dx^2 + h)]$ , we need to replace  $L_2(M, \mathcal{S})$  by  $L_2(\mathbb{R}_{\leq 0} \times \partial M, \mathcal{S})$ . However, this component  $L_2(\mathbb{R}_{\leq 0} \times \partial M, \mathcal{S})$  does not contribute to the class  $\delta \circ \Theta_2 [\rho(D_{\mathbb{R} \times \partial M}, dx^2 + h)]$  in the end and the construction gives a representative identical to that of  $\partial_1(\text{Ind}(D_{M_\infty}))$ .

This finishes the proof. □

*Remark 23.* To deal with manifolds of dimension  $m \not\equiv 0 \pmod{8}$  in the real case, we work with  $\text{Cl}_m$ -linear Dirac operators, cf. [17, Section II.7]. Here  $\text{Cl}_m$  is the standard real Clifford algebra on  $\mathbb{R}^m$  with  $e_i e_j + e_j e_i = -2\delta_{ij}$ . We recall the definition of  $\text{Cl}_m$ -linear Dirac operators in the following. Consider the standard representation  $\ell$  of  $\text{Spin}_m$  on  $\text{Cl}_m$  given by left multiplication. Let  $M$  be the manifold from the beginning of this section and  $P_{\text{spin}}(M)$  be the principal  $\text{Spin}_m$ -bundle, then we define  $\mathfrak{S}$  to be the vector bundle

$$\mathfrak{S} = P_{\text{spin}}(M) \times_\ell \text{Cl}_m.$$

We denote the associated  $\text{Cl}_m$ -linear Dirac operator on  $M_\infty$  by

$$\mathfrak{D} : L^2(M_\infty, \mathfrak{S}) \rightarrow L^2(M_\infty, \mathfrak{S}).$$

Notice that the right multiplication of  $\text{Cl}_m$  on  $\mathfrak{S}$  commutes with  $\ell$ . Moreover,  $\mathfrak{D}$  is invariant under the action of  $\Gamma$ . Then, by the same argument as in Section 3,  $\mathfrak{D}$  defines a higher index class

$$\text{Ind}(\mathfrak{D}) \in \widehat{K}_0(C^*(M, \mathbb{R})^\Gamma \widehat{\otimes} \text{Cl}_m) \cong \widehat{K}_m(C^*(M, \mathbb{R})^\Gamma) \cong K_m(C^*(M, \mathbb{R})^\Gamma),$$

where  $C^*(M, \mathbb{R})^\Gamma$  stands for the  $\Gamma$ -invariant Roe algebra of  $M$  with coefficients in  $\mathbb{R}$ . Moreover,  $\widehat{K}_*$  stands for the  $\mathbb{Z}_2$ -graded  $K$ -theory of  $C^*$ -algebras,  $\widehat{\otimes}$  stands for  $\mathbb{Z}_2$ -graded tensor product, cf. [15, Chapter III]. Notice that for a trivially graded  $C^*$ -algebra  $\mathcal{A}$ , we have

$$\widehat{K}_m(\mathcal{A}) \cong K_m(\mathcal{A}).$$

In the case of a manifold without boundary, i.e.  $\partial M = \emptyset$ , then we can define the local index class of  $\mathfrak{D}$  and denote it by

$$\text{Ind}_L(\mathfrak{D}) \in \widehat{K}_0(C_L^*(M, \mathbb{R})^\Gamma \widehat{\otimes} \text{Cl}_m) \cong \widehat{K}_m(C_L^*(M, \mathbb{R})^\Gamma) \cong K_m(C_L^*(M, \mathbb{R})^\Gamma),$$

whose image under the evaluation map  $\text{ev}_* : K_m(C_L^*(M, \mathbb{R})^\Gamma) \rightarrow K_m(C^*(M, \mathbb{R})^\Gamma)$  is  $\text{Ind}(\mathfrak{D})$ . Moreover, the higher rho invariant is also defined similarly in the real case.

*Remark 24.* Theorem 22 also holds if we use the maximal versions of all the  $C^*$ -algebras in the theorem.

The following corollaries are immediate consequences of Theorem 22. In a similar context, these have already appeared in the work of Lott [19], Botvinnik and Gilkey [7], and Leichtnam and Piazza [18].

**Corollary 25.** *With the same notation as above, if  $\rho(D_{\partial M}, h) \neq 0$ , then there does not exist a  $\Gamma$ -invariant complete Riemannian metric  $g$  on  $M$  with product structure near the boundary  $\partial M$  such that  $g$  has positive scalar curvature and  $g|_{\partial M} = h$ .*

In other words, nonvanishing of the higher rho invariant is an obstruction to extension of the positive scalar curvature metric from the boundary to the whole manifold.

Let  $N$  be a spin manifold without boundary, equipped with a proper cocompact action of a discrete group  $\Gamma$ . Denote by  $\mathcal{R}^+(N)^\Gamma$  the space of all  $\Gamma$ -invariant complete Riemannian metrics of positive scalar curvature on  $N$ . For  $h_0, h_1 \in \mathcal{R}^+(N)^\Gamma$ , we say  $h_0$  and  $h_1$  are path connected in  $\mathcal{R}^+(N)^\Gamma$  if  $h_0$  and  $h_1$  are connected by a smooth path of  $h_t \in \mathcal{R}^+(N)^\Gamma$ ,  $0 \leq t \leq 1$ . More generally, we say  $h_0$  and  $h_1$  are  $\Gamma$ -bordant if there exists a spin manifold  $W$  with a proper cocompact action of  $\Gamma$  such that  $\partial W = N \amalg (-N)$  and  $W$  carries a  $\Gamma$ -invariant complete Riemannian metric  $g$  of positive scalar curvature<sup>6</sup> with  $g|_N = h_0$  and  $g|_{-N} = h_1$ . Here  $-N$  means  $N$  with the opposite orientation. Clearly, if  $h_0$  and  $h_1$  are path connected, then they are  $\Gamma$ -bordant.

**Corollary 26.** *Let  $h_0, h_1 \in \mathcal{R}^+(N)^\Gamma$  as above. If  $\rho(D_N, h_0) \neq \rho(D_N, h_1)$ , then  $h_0$  and  $h_1$  are not  $\Gamma$ -bordant. In particular, this implies that  $h_0$  and  $h_1$  are in different connected components of  $\mathcal{R}^+(N)^\Gamma$ .*

## 5 Stolz's positive scalar curvature exact sequence

In this section, we apply our main theorem (Theorem 22 above) to map the Stolz's positive scalar curvature exact sequence [25] to a long exact sequence of  $K$ -theory of  $C^*$ -algebras.

First, let us recall the Stolz's positive scalar curvature exact sequence.

---

<sup>6</sup>We assume the metric  $g$  has product structure near the boundary.



**Definition 27.** Given a topological space  $X$ , we denote by  $\Omega_n^{\text{spin}}(X)$  the set of bordism classes of pairs  $(M, f)$ , where  $M$  is an  $n$ -dimensional closed spin manifold and  $f : M \rightarrow X$  is a continuous map. Two such pairs  $(M_1, f_1)$  and  $(M_2, f_2)$  are bordant if there is a bordism  $W$  between  $M_1$  and  $M_2$  (with compatible spin structure), and a continuous map  $F : W \rightarrow X$  such that  $F|_{M_i} = f_i$ . Then  $\Omega_n^{\text{spin}}(X)$  is an abelian group with the addition being disjoint union. We call  $\Omega_n^{\text{spin}}(X)$  the  $n$ -dimensional spin bordism of  $X$ .

**Definition 28.** Let  $\text{Pos}_n^{\text{spin}}(X)$  be the bordism group of triples  $(M, f, g)$ , where  $M$  is an  $n$ -dimensional closed spin manifold,  $f : M \rightarrow X$  is a continuous map, and  $g$  is a positive scalar curvature metric on  $M$ . Two such triples  $(M_1, f_1, g_1)$  and  $(M_2, f_2, g_2)$  are bordant if there is a bordism  $(W, F, G)$  such that  $G$  is a positive scalar curvature metric on  $W$  with product structure near  $M_i$  and  $G|_{M_i} = g_i$ , and  $F|_{M_i} = f_i$ .

Then it is clear that forgetting the metric gives a homomorphism

$$\text{Pos}_n^{\text{spin}}(X) \rightarrow \Omega_n^{\text{spin}}(X).$$

**Definition 29.**  $R_n^{\text{spin}}(X)$  is the bordism group of the triples  $(M, f, h)$ , where  $M$  is an  $n$ -dimensional spin manifold (possibly with boundary),  $f : M \rightarrow X$  is a continuous map, and  $h$  is a positive scalar curvature metric on the boundary  $\partial M$ . Two triples  $(M_1, f_1, h_1)$  and  $(M_2, f_2, h_2)$  are bordant if

- (a) there is a bordism  $(V, F, H)$  between  $(\partial M_1, f_1, h_1)$  and  $(\partial M_2, f_2, h_2)$  (considered as triples in  $\text{Pos}_{n-1}^{\text{spin}}(X)$ ),
- (b) and the closed spin manifold  $M_1 \cup_{\partial M_1} V \cup_{\partial M_2} M_2$  (obtained by gluing  $M_1, V$  and  $M_2$  along their common boundary components) is the boundary of a spin manifold  $W$  with a map  $E : W \rightarrow X$  such that  $E|_{M_i} = f_i$  and  $E|_V = F$ .

It is not difficult to see that the three groups defined above fit into the following long exact sequence:

$$\longrightarrow \Omega_{n+1}^{\text{spin}}(X) \longrightarrow R_{n+1}^{\text{spin}}(X) \longrightarrow \text{Pos}_n^{\text{spin}}(X) \longrightarrow \Omega_n^{\text{spin}}(X) \longrightarrow$$

where the maps are defined the obvious way.

Now let  $X$  be a proper metric space equipped with a proper and cocompact isometric action of a discrete group  $\Gamma$ .

**Definition 30.** We denote by  $\Omega_n^{\text{spin}}(X)^\Gamma$  the set of bordism classes of pairs  $(M, f)$ , where  $M$  is an  $n$ -dimensional complete spin manifold (without boundary) such that  $M$  is equipped with a proper and cocompact isometric action of  $\Gamma$  and  $f : M \rightarrow X$  is a  $\Gamma$ -equivariant continuous map. Two such pairs  $(M_1, f_1)$  and  $(M_2, f_2)$  are bordant if there is a bordism  $W$  between  $M_1$  and  $M_2$  (with compatible spin structure), and a  $\Gamma$ -equivariant continuous map  $F : W \rightarrow X$  such that  $F|_{M_i} = f_i$ . Then  $\Omega_n^{\text{spin}}(X)^\Gamma$  is an abelian group with the addition being disjoint union. We call  $\Omega_n^{\text{spin}}(X)^\Gamma$  the  $n$ -dimensional  $\Gamma$ -invariant spin bordism of  $X$ .

Similarly, we can define the groups  $\text{Pos}_n^{\text{spin}}(X)^\Gamma$  and  $R_n^{\text{spin}}(X)^\Gamma$ , which fit into the following exact sequence:

$$\longrightarrow \Omega_{n+1}^{\text{spin}}(X)^\Gamma \longrightarrow R_{n+1}^{\text{spin}}(X)^\Gamma \longrightarrow \text{Pos}_n^{\text{spin}}(X)^\Gamma \longrightarrow \Omega_n^{\text{spin}}(X)^\Gamma \longrightarrow$$

Now suppose  $(M, f) \in \Omega_i^{\text{spin}}(X)^\Gamma$  and denote by  $D_M$  the associated Dirac operator on  $M$ . Recall that there is a local index map (cf. Section 2.3)

$$\text{Ind}_L : K_i^\Gamma(M) \rightarrow K_i(C_L^*(M)^\Gamma).$$

Let  $f_* : K_i(C_L^*(M)^\Gamma) \rightarrow K_i(C_L^*(X)^\Gamma)$  be the map induced by  $f : M \rightarrow X$ . We define the map

$$\text{Ind}_L : \Omega_i^{\text{spin}}(X)^\Gamma \rightarrow K_i(C_L^*(X)^\Gamma), \quad (M, f) \mapsto f_*[\text{Ind}_L(D_M)].$$

Recall that  $K_i(C_L^*(Y)^\Gamma)$  is naturally isomorphic to  $K_i^\Gamma(Y)$  for all finite dimensional simplicial complex  $Y$  equipped with a proper and cocompact isometric action of a discrete group  $\Gamma$  (cf. [28, Theorem 3.2]). Now the well-definedness of the map  $\text{Ind}_L$  follows immediately by using the geometric description of  $K$ -homology groups [5] [6].

Now assume in addition we have a positive scalar curvature metric  $g$  on  $M$ . In other words, we have  $(M, f, g) \in \text{Pos}_i^{\text{spin}}(X)^\Gamma$ . Then we have the map

$$\rho : \text{Pos}_i^{\text{spin}}(X)^\Gamma \rightarrow K_i(C_{L,0}^*(X)^\Gamma), \quad (M, f, g) \mapsto f_*[\rho(D_M, g)].$$

The well-definedness of the map  $\rho$  follows immediately from Theorem 22 above.

Moreover, suppose we have  $(M, f, h) \in R_i^{\text{spin}}(X)^\Gamma$ . By the discussion in Section 3 and the relative higher index theorem (cf. [8][27]), we have the following well-defined homomorphism

$$\text{Ind} : R_i^{\text{spin}}(X)^\Gamma \rightarrow K_i(C^*(X)^\Gamma), \quad (M, f, g) \mapsto f_*[\text{Ind}(D_{M_\infty})].$$

**Theorem 31.** *For all  $n \in \mathbb{N}$ , the following diagram commutes*

$$\begin{array}{ccccccc} \Omega_{n+1}^{\text{spin}}(X)^\Gamma & \longrightarrow & R_{n+1}^{\text{spin}}(X)^\Gamma & \longrightarrow & \text{Pos}_n^{\text{spin}}(X)^\Gamma & \longrightarrow & \Omega_n^{\text{spin}}(X)^\Gamma \\ \downarrow \text{Ind}_L & & \downarrow \text{Ind} & & \downarrow \rho & & \downarrow \text{Ind}_L \\ K_{n+1}(C_L^*(X)^\Gamma) & \longrightarrow & K_{n+1}(C^*(X)^\Gamma) & \xrightarrow{\partial} & K_n(C_{L,0}^*(X)^\Gamma) & \longrightarrow & K_n(C_L^*(X)^\Gamma) \end{array}$$

*Proof.* The commutativities of the first square and the third square follow immediately from the definition.

Let  $(M, f, h) \in R_i^{\text{spin}}(X)^\Gamma$ . Then by Theorem 22, we have

$$\partial(\text{Ind}(D_{M_\infty})) = \rho(D_{\partial M}, h).$$

This shows the commutativity of the second square. □

*Remark 32.* As we shall see in Section 6, we have a natural isomorphism between the following two long exact sequences

$$\begin{array}{ccccccc}
K_{n+1}(D^*(X)^\Gamma) & \longrightarrow & K_{n+1}(D^*(X)^\Gamma/C^*(X)^\Gamma) & \longrightarrow & K_n(C^*(X)^\Gamma) & \longrightarrow & K_n(D^*(X)^\Gamma) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
K_n(C_{L,0}^*(X)^\Gamma) & \longrightarrow & K_n(C_L^*(X)^\Gamma) & \longrightarrow & K_n(C^*(X)^\Gamma) & \longrightarrow & K_{n-1}(C_{L,0}^*(X)^\Gamma)
\end{array}$$

Now if in addition the action of  $\Gamma$  on  $X$  is free, then we have  $\Omega_n^{\text{spin}}(X)^\Gamma = \Omega_n^{\text{spin}}(X/\Gamma)$ ,  $\text{Pos}_n^{\text{spin}}(X)^\Gamma = \text{Pos}_n^{\text{spin}}(X/\Gamma)$ , and  $R_n^{\text{spin}}(X)^\Gamma = R_n^{\text{spin}}(X/\Gamma)$ . Therefore we see that in this case Theorem 31 above recovers [21, Theorem 1.31] of Piazza and Schick. We emphasize that our proof works equally well for both the even and the odd cases. In particular, Theorem 31 holds for all  $n$ .

*Remark 33.* We also have the maximal version of Theorem 31, by replacing all the  $C^*$ -algebras by their maximal versions.

## 6 Roe algebras and localization algebras

In this section, we show the natural isomorphism between the long exact sequence of localization algebras (in Theorem 31) and the long exact sequence of Higson and Roe. In particular, the explicit construction naturally identifies our definition of the higher rho invariant (see Section 2.3.2 above) with the higher rho invariant in Piazza and Schick [21, Section 1.2].

Let  $X$  be a *finite dimensional simplicial complex* equipped with a proper and co-compact isometric action of a discrete group  $\Gamma$ . Consider the following short exact sequences:

$$0 \rightarrow C^*(X)^\Gamma \rightarrow D^*(X)^\Gamma \rightarrow D^*(X)^\Gamma/C^*(X)^\Gamma \rightarrow 0,$$

and

$$0 \rightarrow C_{L,0}^*(X)^\Gamma \rightarrow C_L^*(X)^\Gamma \rightarrow C^*(X)^\Gamma \rightarrow 0.$$

We claim that there are natural homomorphisms

$$\beta_i : K_i(D^*(X)^\Gamma/C^*(X)^\Gamma) \rightarrow K_{i-1}(C_L^*(X)^\Gamma).$$

**Odd case.** That is,  $i$  is an odd integer. Suppose  $u$  is an invertible element in  $D^*(X)^\Gamma/C^*(X)^\Gamma$  with its inverse  $v$ . Let  $U, V \in D^*(X)^\Gamma$  be certain (not necessarily invertible) lifts of  $u$  and  $v$ . Without loss of generality<sup>7</sup>, we assume that  $U$  and  $V$  have finite propagations.

Recall that for each  $n \in \mathbb{N}$ , there exists a  $\Gamma$ -invariant locally finite open cover  $\{Y_{n,j}\}$  of  $X$  such that  $\text{diameter}(Y_{n,j}) < 1/n$  for all  $j$ . Let  $\{\phi_{n,j}\}$  be a  $\Gamma$ -invariant smooth partition of unity subordinate to  $\{Y_{n,j}\}$ . If we write

$$U(t) = \sum_j \left( (1 - (t - n)) \phi_{n,j}^{1/2} U \phi_{n,j}^{1/2} + (t - n) \phi_{n+1,j}^{1/2} U \phi_{n+1,j}^{1/2} \right)$$

---

<sup>7</sup>by choosing approximations of  $U$  and  $V$ , hence approximations of  $u$  and  $v$ , if necessary

for  $t \in [n, n+1]$ . Notice that  $[U, \phi_{n,j}^{1/2}] \in C^*(X)^\Gamma$ . It follows immediately that the path  $U(t)$ ,  $0 \leq t < \infty$ , defines a class in  $K_1(D_L^*(X)^\Gamma/C_L^*(X)^\Gamma)$ . Then the map  $\beta_1$  is defined by

$$\beta_1([u]) = \partial_1[U(t)] \in K_0(C_L^*(X)^\Gamma),$$

where  $\partial_1 : K_1(D_L^*(X)^\Gamma/C_L^*(X)^\Gamma) \rightarrow K_0(C_L^*(X)^\Gamma)$  is the canonical boundary map (cf. Section 2.1) in the six-term  $K$ -theory exact sequence induced by

$$0 \rightarrow C_L^*(X)^\Gamma \rightarrow D_L^*(X)^\Gamma \rightarrow D_L^*(X)^\Gamma/C_L^*(X)^\Gamma \rightarrow 0.$$

It is not difficult to verify that  $\beta_1$  is well-defined.

**Even case.** The even case is parallel to the odd case above. Suppose  $q$  is an idempotent in  $D^*(X)^\Gamma/C^*(X)^\Gamma$ . If  $Q$  is a lift of  $q$ , then we define

$$Q(t) = \sum_i \left( (1 - (t - n))\phi_{n,j}^{1/2} Q \phi_{n,j}^{1/2} + (t - n)\phi_{n+1,j}^{1/2} Q \phi_{n+1,j}^{1/2} \right)$$

for  $t \in [n, n+1]$  with the same  $\phi_{n,j}$  as above. Now the map  $\beta_0$  is defined by

$$\beta_0([q]) = \partial_0[Q(t)] \in K_0(C_{L,0}^*(X)^\Gamma),$$

where  $\partial_0 : K_0(D_L^*(X)^\Gamma/C_L^*(X)^\Gamma) \rightarrow K_1(C_L^*(X)^\Gamma)$  is the canonical boundary map in the six-term  $K$ -theory exact sequence induced by

$$0 \rightarrow C_L^*(X)^\Gamma \rightarrow D_L^*(X)^\Gamma \rightarrow D_L^*(X)^\Gamma/C_L^*(X)^\Gamma \rightarrow 0.$$

Now we turn to the maps

$$\alpha_i : K_i(D^*(X)^\Gamma) \rightarrow K_{i-1}(C_{L,0}^*(X)^\Gamma).$$

Suppose  $u$  (resp.  $q$ ) is an invertible element (resp. idempotent) in  $D^*(X)^\Gamma$ . In this case, we can simply choose  $U = u$  (resp.  $Q = q$ ). Let  $U(t)$  (resp.  $Q(t)$ ) be as above. Now we apply the construction of the index map  $\partial_1$  (resp.  $\partial_0$ ) in Section 2.1 to  $U(t)$  (resp.  $Q(t)$ ). One immediately sees that the resulting element is an idempotent (resp. invertible element) in

$$(C_{L,0}^*(X)^\Gamma)^+.$$

We define  $\alpha_1(u)$  (resp.  $\alpha_0(q)$ ) to be this element in  $K_0(C_{L,0}^*(X)^\Gamma)$  (resp.  $K_1(C_{L,0}^*(X)^\Gamma)$ ). Again it is not difficult to see that  $\alpha_i$  is well-defined.

**Proposition 34.** *The homomorphisms*

$$\alpha_i : K_i(D^*(X)^\Gamma) \rightarrow K_{i-1}(C_{L,0}^*(X)^\Gamma)$$

and

$$\beta_i : K_i(D^*(X)^\Gamma/C^*(X)^\Gamma) \rightarrow K_{i-1}(C_L^*(X)^\Gamma)$$

are isomorphisms. Moreover,  $\alpha_i$  and  $\beta_i$  are natural in the sense that the following diagram commutes.

$$\begin{array}{ccccccc} K_{i+1}(D^*(X)^\Gamma) & \longrightarrow & K_{i+1}(D^*(X)^\Gamma/C^*(X)^\Gamma) & \longrightarrow & K_i(C^*(X)^\Gamma) & \longrightarrow & K_i(D^*(X)^\Gamma) \\ \downarrow \alpha_{i+1} & & \downarrow \beta_{i+1} & & \downarrow \cong & & \downarrow \alpha_i \\ K_i(C_{L,0}^*(X)^\Gamma) & \longrightarrow & K_i(C_L^*(X)^\Gamma) & \longrightarrow & K_i(C^*(X)^\Gamma) & \longrightarrow & K_{i-1}(C_{L,0}^*(X)^\Gamma) \end{array}$$

*Proof.* The commutativity of the diagram follows by the definitions of the maps  $\alpha_i$  and  $\beta_i$ .

We first prove that  $\beta_i$  is an isomorphism. Notice that we have Mayer-Vietoris sequences for  $K$ -theory of both  $D^*(X)^\Gamma/C^*(X)^\Gamma$  and  $C_L^*(X)^\Gamma$  (cf. [23, Chapter 5] and [28]). It is trivial to verify that  $\beta_i$  is an isomorphism when  $X$  is 0-dimensional (i.e. consisting of discrete points). By using Mayer-Vietoris sequence and the five lemma, the general case follows from induction on the dimension of  $X$ .

Now it follows from the five lemma that  $\alpha_i$  is an isomorphism. This finishes the proof. □

*Remark 35.* In fact both  $K_{i+1}(D^*(X)^\Gamma/C^*(X)^\Gamma)$  and  $K_i(C_L^*(X)^\Gamma)$  are naturally identified with the  $K$ -homology group  $K_i^\Gamma(X)$  (cf. [23, Chapter 5] and [28]). The map  $\beta_i$  equates these two natural identifications.

*Remark 36.* By following the construction of the isomorphism  $\alpha_i$ , one immediately sees that our definition of the higher rho invariant (see Section 2.3.2 above) is equivalent to that of Piazza and Schick [21, Section 1.2].

*Remark 37.* We also have the maximal version of Proposition 34, by replacing all the  $C^*$ -algebras by their maximal versions.

## References

- [1] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. *Bull. London Math. Soc.*, 5:229–234, 1973.
- [2] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.
- [3] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69:422–433, 1963.
- [4] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and  $K$ -theory of group  $C^*$ -algebras. In  *$C^*$ -algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.
- [5] P. Baum and R. G. Douglas.  $K$  homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.
- [6] P. Baum, N. Higson, and T. Schick. A geometric description of equivariant  $K$ -homology for proper actions. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 1–22. Amer. Math. Soc., Providence, RI, 2010.
- [7] B. Botvinnik and P. B. Gilkey. The eta invariant and metrics of positive scalar curvature. *Math. Ann.*, 302(3):507–517, 1995.

- [8] U. Bunke. A  $K$ -theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, 303(2):241–279, 1995.
- [9] A. Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.
- [10] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.
- [11] A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29(3):345–388, 1990.
- [12] N. Higson and J. Roe. Mapping surgery to analysis. I. Analytic signatures. *K-Theory*, 33(4):277–299, 2005.
- [13] N. Higson and J. Roe. Mapping surgery to analysis. II. Geometric signatures. *K-Theory*, 33(4):301–324, 2005.
- [14] N. Higson and J. Roe. Mapping surgery to analysis. III. Exact sequences. *K-Theory*, 33(4):325–346, 2005.
- [15] M. Karoubi. *K-theory*. Springer-Verlag, Berlin, 1978. An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226.
- [16] G. Kasparov. Equivariant  $KK$ -theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.
- [17] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [18] E. Leichtnam and P. Piazza. On higher eta-invariants and metrics of positive scalar curvature. *K-Theory*, 24(4):341–359, 2001.
- [19] J. Lott. Higher eta-invariants. *K-Theory*, 6(3):191–233, 1992.
- [20] A. S. Miščenko. Infinite-dimensional representations of discrete groups, and higher signatures. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:81–106, 1974.
- [21] P. Piazza and T. Schick. Rho-classes, index theory and Stolz’s positive scalar curvature sequence. 10 2012.
- [22] J. Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Mem. Amer. Math. Soc.*, 104(497):x+90, 1993.
- [23] J. Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.
- [24] J. Rosenberg.  $C^*$ -algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.

- [25] S. Stolz. Positive scalar curvature metrics—existence and classification questions. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 625–636, Basel, 1995. Birkhäuser.
- [26] S. Weinberger. Higher  $\rho$ -invariants. In *Tel Aviv Topology Conference: Rothenberg Festschrift (1998)*, volume 231 of *Contemp. Math.*, pages 315–320. Amer. Math. Soc., Providence, RI, 1999.
- [27] Z. Xie and G. Yu. A relative higher index theorem, diffeomorphisms and positive scalar curvature. submitted, available at [arxiv.org/abs/1204.3664](https://arxiv.org/abs/1204.3664), 2012.
- [28] G. Yu. Localization algebras and the coarse Baum-Connes conjecture. *K-Theory*, 11(4):307–318, 1997.
- [29] G. Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)*, 147(2):325–355, 1998.
- [30] G. Yu. A characterization of the image of the Baum-Connes map. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 649–657. Amer. Math. Soc., Providence, RI, 2010.